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On Solvability of Optimization Problem for Elastic Oscillations with Multipoint Sources of Control

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Abstract. In the paper we investigate the optimal control problem for elastic oscillation under multipoint influences of external forces when oscillation process is described by Fredholm integro-differential equation. Sufficient conditions for unique solvability of nonlinear optimization problem were found and the algorithm for constructing complete solution to this problem was developed.

INTRODUCTION

In practice, applied problems described by integro-differential equations are frequently happened [1, 2, 3]. Control problems for processes described by partial integro-differential equations of parabolic or hyperbolic types, when control is linearly included in the equation, were investigated in [4, 5, 6, 7, 8, 9, 10]. In fact a lot of applied problems are nonlinear. Nonlinear problems refer to the little studied area of optimal control theory.

In this paper we consider the control problem for oscillation process when the external force is centered in several fixed points. The actions of external forces at fixed points are mathematically described by Dirac delta function, which is a singular generalized function; in particular, its filtering property is used. In this study sufficient conditions for unique solvability of nonlinear optimization problem for elastic oscillations with multipoint external influences are found. The algorithm was developed for constructing the complete solution of the nonlineraly optimization problem in the form of triplet $\Big(u^0(t),\;\; V^0(t,x),\;\; J[u^0(t)]\Big)$, where $u^0(t)$ is the optimal control, $V^0(t,x)$ is the optimal process, and $J[u^0(t)]$ is the minimal value of the functional.

This control problem is the continuation of [11], and further we will use the same notations given in this work.

PROBLEM FORMULATION

We consider the problem of nonlinear optimization for elastic oscillations with multipoint controls, where it is required to minimize the following generalized quadratic functional

$$J[u_1(t),...,u_m(t)] = \int_0^1 \Big([V(T,x)-\xi_1(x)]^2 + [V_t(T,x)-\xi_2(x)]^2\Big)dx + \beta\int_0^T \sum_{k=1}^{m} p_k^2[t,u_k(t)]dt, \quad \beta>0, \tag{1}$$

on the set of solutions to following boundary value problem

$$V_{tt} = V_{xx} + \lambda\int_0^T K(t,\tau)v(\tau,x)d\tau + \sum_{k=1}^{m} g_k(x)\delta(x-x_k)f_k[t,u_k(t)], \quad 0<x<1, \;\; 0<t<T,$$

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$$V(0,x)=\psi_1(x),\ \ V_t(0,x)=\psi_2(x),\ \ 0<x<1, \tag{2}$$
$$V_x(t,0)=0,\ \ V_x(t,1)+\alpha V(t,1)=0,\ \ 0<t\le T<\infty.$$

Here function $V(t,x)$ describes the state of an elastic thread with a length equal to one, which oscillates under influences of external disturbing forces $g_k(x)f_k[t,u_k(t)],\ \ k=1,...m,$ and apply respectively to internal points $x_1,...,x_m$ of interval $(0,1)$; $\xi_1(x),\xi_1(x)\in H(0,1),\ \ \psi_1(x)\in H_1(0,1),\ \ \psi_2(x)\in H(0,1), p_k[t,u_k(t)]\in H(0,T),\ f_k[t,u_k(t)]\in H(0,T)$ are given functions, and functions $f_k[t,u_k(t)]$ are nonlinear with respect to functional variable $u_k(t)$ and they are monotonous functions, i.e.

$$\frac{\partial f_k[t,u_k(t)]}{\partial u_k(t)}\neq 0,\ \ \forall t\in[0,T]. \tag{3}$$

Kernel $K(t,\tau)$ is defined in the region $D=\{0\le t\le T,\ 0\le\tau\le T\}$ and satisfies the following condition

$$\int_0^1\int_0^1 K^2(t,\tau)d\tau dt=K_0<\infty; \tag{4}$$

i.e. $K(t,\tau)$ is an element of Hilbert space $H(D)$. $H(Y)$ is a Hilbert space of quadratically summable functions defined on the set Y; $H_1(Y)$ is a Sobolev space of the first order. Given functions $g_k(x)$ and its generalized derivatives are finite on interval $(0,1)$ for each fixed $k=1,...,m,\ \ \delta(x-x_k)$ is a singular generalized Dirac delta function; T is a fixed moment of time, α is a positive constant, λ is a parameter.

The solution to boundary value problem (2) we will seek in the form

$$V(t,x)=\sum_{n=1}^{\infty}V_n(t)z_n(x),\ \ V_n(t)=\int_0^1 V(t,x),z_n(x)dx, \tag{5}$$

where

$$z_n(x)=\sqrt{\frac{2(\lambda_n^2+\alpha^2)}{\lambda_n^2+\alpha^2+\alpha}}\cos\lambda_n x,\quad n=1,2,3,...,$$

are defined as solution to following boundary value problem

$$z_n''(x)+\lambda_n^2 z_n(x)=0,\quad z_n'(0)=0,\quad z_n'(1)+\alpha z_n(1)=0.$$

These functions form the complete orthonormal system of eigenfunctions in the Hilbert space $H(0,1)$ and the corresponding eigenvalues λ_n are solutions to equation $\lambda\tan\lambda=\alpha$ and satisfy the conditions $\lambda_n<\lambda_{n+1},\ \ \lim_{n\to\infty}\lambda_n=\infty,\ \ (n-1)\pi<\lambda_n<\frac{\pi}{2}(2n-1),\ \ n=1,2,3,....$

Fourier coefficients $V_n(t)$ are defined as solution to following linear inhomogeneous Fredholm integral equation of the second kind

$$V_n(t)=\int_0^T K_n(t,s)V_n(s)ds+a_n(t),\ \ n=1,2,3,..., \tag{6}$$

where

$$K_n(t,s)=\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)K(\tau,s)d\tau,\ \ K(0,s)=0,$$

$$a_n(t)=\psi_{1n}\cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}\sin\lambda_n t+\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)\sum_{k=1}^{m}g_k(x_k)z_n(x_k)f_k[\tau,u_k(\tau)]d\tau.$$

We find the solution to integral equation (6) by the following formula [12]:

$$V_n(t)=\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds+a_n(t), \tag{7}$$

where

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda_n^{i-1} K_{n,i}(t,s),\ n = 1,2,3,... \tag{8}$$

is the resolvent of kernel $K_{n,1}(t,s) \equiv K_n(t,s)$. Iterated kernels $K_{n,i}(t,s)$ are defined by formulas $K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta,\ i = 1,2,3,...$ and the following estimate holds

$$\int_0^T R_n^2(t,s,\lambda)ds \le \frac{K_0 T}{(\lambda_n - |\lambda|\sqrt{K_0 T^2})^2}. \tag{9}$$

The radius of convergence of Neumann series (8) is determined by inequality

$$|\lambda| < \frac{\lambda_n}{\sqrt{K_0 T^2}},$$

for the parameter λ and for each fixed $n = 1,2,3,....$

It follows from (10) that the convergence radius of Neumann series expand when n is growing. However the Neumann series converges for the values of the parameter λ satisfying inequality

$$|\lambda| < \frac{\lambda_1}{\sqrt{K_0 T^2}}, \tag{10}$$

for every $n = 1,2,3,....$

Note that this interval can be expanded by decreasing the value of K_0.

Thus the solution to boundary value problem (2) is found by following formula

$$V(t,x) = \sum_{n=1}^{\infty} \Big\{ \psi_n(t,\lambda) + \int_0^T \varepsilon_n(t,\tau,\lambda) \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[\tau, u_k(\tau)] d\tau \Big\} z_n(x),$$

where

$$\varepsilon_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\tau) + \frac{\lambda}{\lambda_n}\int_\tau^T R_n(t,s,\lambda)\sin\lambda_n(s-\tau)ds, & 0 \le \tau \le t, \\ \frac{\lambda}{\lambda_n}\int_\tau^T R_n(t,s,\lambda)\sin\lambda_n(s-\tau)ds, & t \le \tau \le T, \end{cases}$$

$$\psi_n(t,\lambda) = \psi_{1n}\left[\cos\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\cos\lambda_n s ds\right] + \frac{1}{\lambda_n}\psi_{2n}\left[\sin\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds\right].$$

Computed solution $V(t,x)$ and its generalized derivative

$$V_t(t,x) = \sum_{n=1}^{\infty} \Big\{ \psi'_{tn}(t,\lambda) + \int_0^T \varepsilon'_{tn}(t,\tau,\lambda) \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[\tau, u_k(\tau)] d\tau \Big\} z_n(x),$$

$$\varepsilon'_{tn}(t,\tau,\lambda) = \begin{cases} \lambda_n\cos\lambda_n(t-\tau) + \frac{\lambda}{\lambda_n}\int_\tau^T R'_{tn}(t,s,\lambda)\sin\lambda_n(s-\tau)ds, & 0 \le \tau \le t, \\ \frac{\lambda}{\lambda_n}\int_\tau^T R'_{tn}(t,s,\lambda)\sin\lambda_n(s-\tau)ds, & t \le \tau \le T, \end{cases}$$

$$\int_0^T R'^2_{nt}(t,s,\lambda)ds = \frac{K_0 T \lambda_n^2}{(\lambda_n - |\lambda|\sqrt{K_0 T^2})^2},$$

$$\psi'_{tn}(t,\lambda) = \psi_{1n}\left[-\lambda_n\sin\lambda_n t + \lambda\int_0^T R'_{nt}(t,s,\lambda)\cos\lambda_n s ds\right] + \frac{1}{\lambda_n}\psi_{2n}\left[\lambda_n\cos\lambda_n t + \lambda\int_0^T R'_{nt}(t,s,\lambda)\sin\lambda_n s ds\right],$$

are elements of the space $H(Q),\ \ Q = \{0 < x < 1,\ \ 0 < t \le T\}$.

THE INCREMENT OF THE FUNCTIONAL AND OPTIMALITY CONDITION

According to monotonicity condition (3), each $u(t) = \{u_1(t), ..., u_m(t)\}$ vector-control uniquely determines the function $V(t, x)$. Taking into account this circumstance, we calculate the increment of functional $J[u(t)]$

$$\Delta J[u(t)] = -\int_0^T \Delta\Pi[t, V(t, x), \omega(t, x), u(t)]dt + \int_0^1 \{\Delta V^2(T, x) + \Delta V_t^2(T, x)\}dx,$$

where

$$\Delta\Pi[t, \cdot, u(t)] = \Pi[t, \cdot, u(t) + \Delta u(t)] - \Pi[t, \cdot, u(t)],$$

$$\Pi[t, V(t, x), \omega(t, x), u(t)] = \sum_{k=1}^{m} g_k(x_k)\omega(t, x_k) f_k[t, u_k(t)] - \beta \sum_{k=1}^{m} p_k^2[t, u_k(t)]. \tag{11}$$

Here function $\omega(t, x)$ is a solution to following adjoint boundary value problem:

$$\begin{aligned} \omega_{tt} = \omega_{xx} + \lambda \int_0^T K(\tau, t)\omega(\tau, x)d\tau, \quad 0 < x < 1, \quad 0 \le t < T,& \\ \omega(T, x) = -2[V_t(T, x) - \xi_2(x)],& \\ \omega_t(T, x) = 2[V(T, x) - \xi_1(x)],& \\ \omega_x(t, 0) = 0, \quad \omega_x(t, 1) + \alpha\omega(t, 1) = 0.& \end{aligned} \tag{12}$$

It is not difficult to prove that following assertion holds (maximum principle): In order for the control $u(t) = \{u_1(t), ..., u_m(t)\} \in H^m(0, T) = H(0, T) \times ... \times H(0, T)$ to be optimal, it is necessary and sufficient that the relation

$$\Pi[t, V(t, x), \omega(t, x), u(t)] = \sup_{u \in F} \Pi[t, V(t, x), \omega(t, x), u]$$

is satisfied for almost all $t \in (0, T)$ in the acceptable region F.

Since the acceptable region F is the open set, then as the consequence of maximum principle for optimality of controls $u_1(t), ..., u_m(t)$, we obtain the following relations

$$2\beta \frac{p_k[t, u_k(t)] p_{ku_k}[t, u_k(t)]}{f_{ku_k}[t, u_k(t)]} = g_k(x_k)\omega(t, x_k), \tag{13}$$

$$\prod_{k=1}^{m} (-1)^k f_{ku_k}[t, u_k(t)] \frac{p_k[t, u_k(t)] p_{ku_k}[t, u_k(t)]}{f_{ku_k}[t, u_k(t)]} = g_k(x_k)\omega(t, x_k) > 0, \quad k = 1, ...m, \tag{14}$$

which are called *optimality conditions*. Here

$$f_{ku_k}[t, u_k(t)] = \frac{\partial f_k[t, u_k(t)]}{\partial u_k}, \quad p_{ku_k}[t, u_k(t)] = \frac{\partial p_k[t, u_k(t)]}{\partial u_k}.$$

SYSTEM OF NONLINEAR INTEGRAL EQUATIONS OF OPTIMAL CONTROLS

According to optimality condition (13), it is necessary to define the function $\omega(t, x)$ as a solution to the boundary value problem (12), which is solved similarly to boundary-value problem (2). It is determined by the formula

$$\omega(t, x) = -2 \sum_{n=1}^{\infty} \left(-G_n^*(T, t, \lambda)h_n + \int_0^T G_n^*(T, t, \lambda)E_n(T, \tau, \lambda) \sum_{k=1}^{m} g_k(x_k)z_n(x_k) f_k[\tau, u_k(\tau)]d\tau \right) z_n(x),$$

where a symbol $*$ denotes the sign of transposition,

$$\begin{aligned} G_n[T,t,\lambda] &= \Big(G_{1n}[T,t,\lambda],\ \ G_{2n}[T,t,\lambda]\Big),\\ E_n[T,\tau,\lambda] &= \Big(\varepsilon_n[T,\tau,\lambda],\ \ \acute{\varepsilon}_{tn}[T,\tau,\lambda]\Big),\\ h_n &= \Big(\xi_{1n}-\psi_n(T,\lambda),\ \ \xi_{2n}-\acute{\psi}_{tn}(T,\lambda)\Big), \end{aligned}$$

$$\begin{aligned} G_{1n}[T,t,\lambda] &= \sin\lambda_n(T-t)+\lambda\int_0^T P_n(s,t,\lambda)\sin\lambda_n(T-t)ds,\\ G_{2n}[T,t,\lambda] &= \cos\lambda_n(T-t)+\lambda\int_0^T P_n(s,t,\lambda)\cos\lambda_n(T-t)ds, \end{aligned}$$

$P_n(s,t,\lambda)$ is a resolvent which appears in solving to adjoint boundary value problem and for this resolvent following estimates hold

$$\int_0^T P_n^2(s,t,\lambda)ds \le \frac{K_0T}{(\lambda_n-|\lambda|\sqrt{K_0T^2})^2}.$$

According to condition (13) with respect to optimal control $u^0(t)$, we obtain the following system of nonlinear integral equations

$$\begin{aligned} \beta\frac{p_k[t,u_k(t)]p_{ku_k}[t,u_k(t)]}{f_{ku_k}[t,u_k(t)]} + \sum_{n=1}^{\infty} g_k(x_k)z_n(x_k)G_n^*(T,t,\lambda)\int_0^T E_n(T,\tau,\lambda)\sum_{k=1}^{m} g_k(x_k)z_n(x_k)f_k[\tau,u_k(\tau)]d\tau = \\ = \sum_{n=1}^{\infty} g_k(x_k)z_n(x_k)G_n^*(T,t,\lambda)h_n,\ \ k=1,...,m. \qquad (15) \end{aligned}$$

Note that the solution to this system should also satisfy the optimality conditions (14). We assume that functions $f_k[t,u_k(t)]$ satisfy the condition (14). According to the method of work [13], we introduce the notation

$$\beta\frac{p_k[t,u_k(t)]p_{ku_k}[t,u_k(t)]}{f_{ku_k}[t,u_k(t)]} = \sigma_k(t),\ \ k=1,...,m. \qquad (16)$$

According to condition (14) from (16) the functions $u_k(t)$ are uniquely determined, i.e.

$$u_k(t) = \varphi_k[t,\sigma_k(t),\beta],\ \ k=1,...,m. \qquad (17)$$

Taking into account (16) and (17), we rewrite system (15) in vector form

$$\begin{aligned} \sigma(t) + \sum_{n=1}^{\infty} K_n[x_1,..,x_m]G_n^*(T,t,\lambda)\int_0^T E_n(T,\tau,\lambda)\sum_{k=1}^{m} K_n^*[x_1,..,x_m]f[\tau,\varphi(\tau,\sigma(\tau),\beta)]d\tau = \\ = \sum_{n=1}^{\infty} K_n[x_1,..,x_m]G_n^*(T,t,\lambda)h_n, \end{aligned}$$

where

$$\begin{aligned} \sigma(t) &= \Big(\sigma_1(t),...,\sigma_m(t)\Big),\\ K_n[x_1,..,x_m] &= \Big(g_1(x_1)z_n(x_1),...,g_m(x_m)z_n(x_m)\Big),\\ \varphi[\tau,\sigma(\tau),\beta] &= \Big(\varphi_1[\tau,\sigma_1(\tau),\beta],...,\varphi_m[\tau,\sigma_m(\tau),\beta]\Big),\\ f[\tau,\varphi(\tau,\sigma(\tau),\beta)] &= \Big(f_1[\tau,\varphi_1(\tau,\sigma_1(\tau),\beta)],...,f_m[\tau,\varphi_m(\tau,\sigma_m(\tau),\beta)]\Big). \end{aligned}$$

Further by direct calculations following lemmas are proved.

Lemma 1 *The m^{th} dimensional vector-function*

$$h(t) = \sum_{n=1}^{\infty} K_n[x_1, .., x_m] G_n^*(T, t, \lambda) h_n = \Big(h_1(t), ...h_m(t)\Big)$$

is an element of the space $H^m(0, T) = H(0, T) \times ... \times H(0, T)$.

Lemma 2 *The operator K maps the spaces $H^m(0, T)$ into itself, i.e.*

$$K[\sigma(t)] = \sum_{n=1}^{\infty} K_n[x_1, .., x_m] G_n^*(T, t, \lambda) \int_0^T E_n(T, \tau, \lambda) \sum_{k=1}^{m} K_n^*[x_1, .., x_m] f[\tau, \varphi(\tau, \sigma(\tau), \beta)] d\tau$$

is an element of the space $H^m(0, T)$ for any $\sigma(t) \in H^m(0, T)$.

Lemma 3 *Suppose that the functions $f_k[t, u_k(t)]$ and $\varphi_k[t, \sigma_k(t), \beta]$ satisfy the following Lipschitz conditions:*

$$\Big|f_k[t, u_k(t)] - f_k[t, \tilde{u}_k(t)]\Big| \le f_{k0}\Big|u_k(t) - \tilde{u}_k(t)\Big|, \quad f_{k0} > 0,$$
$$\Big|\varphi_k[t, \sigma_k(t), \beta] - \varphi_k[t, \tilde{\sigma}_k(t), \beta]\Big| \le \varphi_{k0}\Big|\sigma_k(t) - \tilde{\sigma}_k(t)\Big|, \quad \varphi_{k0}(\beta) > 0.$$

Then if the condition

$$\gamma = 8T \sum_{k=1}^{m} \tilde{g}_{k0}^2 \Big(\frac{1}{\lambda_n^2} + \frac{1}{6}\Big)\Big(1 + \frac{\lambda^2 K_0 T^2}{(\lambda_1 - |\lambda| \sqrt{K_0 T^2})^2}\Big) \overline{f}_0 \overline{\varphi}_0(\beta) < 1, \quad f_0 = \max(f_{10}, ..., f_{m0}), \quad \varphi_0(\beta) = \max(\varphi_{10}(\beta), ..., \varphi_{m0}(\beta))$$

holds, the operator K is contractive.

For unique solvability of nonlinear optimization problem for elastic oscillations with multipoint controls, the following theorem is true.

Theorem 4 *Suppose that conditions of optimization problem (1)-(4), conditions of Lemmas 1-3 and the condition (14) are satisfied, then operator equation $\sigma = K[\sigma] + h$ has an unique solution in space $H^m(0, T)$.*

Proof. As $H^m(0, T)$ is complete metric space and K is the operator mapping into itself, according to the well-known theorem on contracting mappings [14], the operator equation $\sigma = K[\sigma] + h$ has an unique solution in the space $H^m(0, T)$ that can be found by the method of successive approximations

$$\sigma_n = K[\sigma_{n-1}] + h, \quad n = 1, 2, 3, ...,$$

where $\sigma_0(t)$ is an arbitrary element of the space $H^m(0, T)$. The exact solution $\sigma^0(t) = \lim\limits_{n\to\infty} \sigma_n(t)$ and its approximations satisfy the following relation:

$$\|\sigma(t) - \sigma_n(t)\|_{H^m(0,T)} \le \frac{\gamma^n}{1 - \gamma} \|K[\sigma_0(t)] + h(t) - \sigma_0(t)\|_{H^m(0,T)}.$$

Substituting the computed solution $\sigma^0(t) = \Big(\sigma_1^0(t), ..., \sigma_m^0(t)\Big)$ in (17), we obtain the optimal control

$$u_k^0(t) = \varphi_k[t, \sigma_k^0(t), \beta], \quad k = 1, ..., m,$$

which is the solution to system of nonlinear integral equations (15) and satisfying conditions (14).

Next we find the optimal process

$$V^0(t, x) = \sum_{n=1}^{\infty} \Big\{\psi_n(t, \lambda) + \int_0^T \varepsilon_n(t, \tau, \lambda) \sum_{k=1}^{m} g_k(x_k) z_n(x_k) f_k[\tau, u_k^0(\tau)] d\tau\Big\} z_n(x),$$

and calculate the minimum value of the functional

$$J[u^0(t)] = \int_0^1 \Big([V^0(T, x) - \xi_1(x)]^2 + [V_t^0(T, x) - \xi_2(x)]^2\Big) dx + \beta \int_0^T \sum_{k=1}^{m} p_k^2[t, u_k^0(t)] dt. \quad \beta > 0.$$

The found triplet $\Big(u^0(t), \quad V^0(t, x), \quad J[u^0(t)]\Big)$ is a complete solution to the nonlinear optimization problem.

REFERENCES

[1] V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations* (Dover, New York, 1959).

[2] V. S. Vladimirov, Tr. MIAN **61**, (1982). (in russian)

[3] A. I. Egorov, *Optimal control of thermal and diffusion processes* (Nauka, Moscow, 1978). (in russian)

[4] E. W. Sachs and A. K. Strauss, Appl. Numer. Math. **58**:11, 1687–1703 (2008). (in russian)

[5] A. Kowalewski, Automatyka **15**, 53–65 (2011).

[6] J. Thorwe and S. Bhalekar, American J. Comput. Appl. Math. **2**:3, 101–104 (2012).

[7] As. Zh. Khurshudyan, Arch. Contol. Sci. **24(60)**:1, 5–25 (2014).

[8] A. Kerimbekov and U. E. Duyshenalieva, Vestnik KRSU, **16**:5, 51–57 (2016). (in russian)

[9] U. E. Duyshenalieva, Vestnik KRSU **16**:9, 7–11 (2016). (in russian)

[10] A. Kerimbekov and U. E. Duyshenalieva, Problemi avtomatiki i upravleniya **2**, 57–62 (2016). (in russian)

[11] A. Kerimbekov, E. Abdyldaeva and U. E. Duyshenalieva, International Journal of Pure and Applied Mathematics **113**:4, 87–101 (2017). (in russian)

[12] M. V. Krasnov, *Integral equations* (Nauka, Moscow, 1975). (in russian)

[13] A. Kerimbekov, *Nonlinear optimal control of the linear systems with distributed parameters* (Texts and Monographs in Physical and Mathematical Sciences, Ilim, Bishkek, 2003). (in russian)

[14] L. A. Lusternik and V. I. Sobolev, *Elements of functional analysis* (Nauka, Moscow, 1965). (in russian)